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# Axial-Vector Duality as a Mirror Symmetry

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## Abstract

We study  $N = 2$  supersymmetric  $SU(2)/U(1)$  and  $SL(2, R)/U(1)$  gauged Wess-Zumino-Witten models. It is shown that the vector gauged model is transformed to the axial gauged model by a mirror transformation. Therefore the vector gauged model and the axial gauged model are equivalent as  $N = (2, 2)$  superconformal field theories. In the  $SL(2, R)/U(1)$  model, it is known that axial-vector duality relates a background with a singularity to that without a singularity. Implications of the equivalence of these two models to space-time singularities are discussed.

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# 1 Introduction

Target-space duality is one of the most striking properties in string theory [1]. It relates backgrounds with different geometries that are equivalent as conformal field theories. This is a characteristic property of string theory, which is not observed in theories based on point particles. As in the simplest example,  $R \leftrightarrow 1/R$  duality of a circle compactified model [2], target space duality generally relates physics at small scales to that at large scales. Hence it plays a crucial role in understanding geometries at short distance, especially issues of singularities, which is inevitable in general relativity.

From this point of view, it is important to study axial-vector duality in gauged Wess-Zumino-Witten models, since these models are related to backgrounds with singularities such as black holes [3]. The most interesting example is a  $SL(2, R)/U(1)$  model where axial-vector duality relates a background with a singularity to that without a singularity. This is a signal that singularities are smeared out in string theory.

Recently, another type of duality, mirror symmetry, is studied mainly in relation to Calabi-Yau manifolds [4]. This is a duality between  $N = (2, 2)$  superconformal field theories. In refs. [5, 6], it is pointed out that axial-vector duality in  $N = 2$  supersymmetric gauged Wess-Zumino-Witten models is nothing but mirror symmetry in some cases. This opens a new possibility to study axial-vector duality from a different standpoint.

In this paper, we present another and more direct method than refs. [5, 6] to describe the equivalence between axial-vector duality and mirror symmetry in the  $SU(2)/U(1)$  model. Then we apply this method to the  $SL(2, R)/U(1)$  model. It is shown that the vector gauged model which corresponds to a singular background is a mirror partner of the axial gauged model which corresponds to a non-singular background. Although this result is already pointed out in ref. [6], our method enables us to clarify the reason why the mirror transformation relates the singular background to the non-singular background.

We also discuss implications of the result to space-time singularities. In fact, it seems that the result leads to a puzzle in the following sense. Theories which are related by mirror symmetry should have the same structures in terms of  $N = (2, 2)$  superconformal field theories. Especially, they should have the same chiral ring. However, in ref. [7], it is suggested that the chiral ring is intimately connected with the existence of space-time singularities in the context of twisted  $N = 2$  gauged Wess-Zumino-Witten models. If this is the case, it seems that a model with a singularity cannot be a mirror partner of a model without a singularity, since they might have different chiral ring. We discuss a possible resolution of this puzzle.

## 2 Axial-vector duality and mirror transformation

In this section, let us recapitulate axial-vector duality and mirror symmetry in the  $N = 2$  gauged Wess-Zumino-Witten model. The action of the Wess-Zumino-Witten model [8] on a group manifold  $G$  at level  $k$  is

$$S(g) = -\frac{k}{8\pi} \int_{\Sigma} d^2\sigma \sqrt{h} h^{ij} \text{Tr}(g^{-1} \partial_i g g^{-1} \partial_j g) - ik\Gamma, \quad (2.1)$$

$$\Gamma = \frac{1}{12\pi} \int_B d^3\sigma \epsilon^{ijk} \text{Tr}(g^{-1} \partial_i g g^{-1} \partial_j g g^{-1} \partial_k g), \quad (2.2)$$

where  $\Sigma$  is a Riemann surface,  $h_{ij}$  is a metric on  $\Sigma$ ,  $g$  is a map from  $\Sigma$  to  $G$ ,  $B$  is any three-manifold whose boundary is  $\Sigma$ .

Due to the Polyakov-Wiegmann formula [9]

$$S(gh) = S(g) + S(h) - \frac{k}{2\pi} \int_{\Sigma} d^2z \text{Tr}(g^{-1} \partial_z g \partial_{\bar{z}} h h^{-1}), \quad (2.3)$$

the action (2.1) is invariant under  $G_L \times G_R$  transformation,  $g \rightarrow v(\bar{z}) g w(z)$ . Not all of this symmetry can be gauged because of anomaly. If we want to gauge abelian subgroup  $H$ , two types of gauging is allowed [10]. One is vector gauging,

$$g \rightarrow u^{-1} g u, \quad (2.4)$$

the other is axial gauging,

$$g \rightarrow u g u. \quad (2.5)$$

By introducing gauge fields  $A_i$  which transform as  $A_i \rightarrow A_i + u^{-1} \partial_i u$ , the action of the gauged Wess-Zumino-Witten model can be written as

$$S_{V/A}(g, A) = S(g) + \frac{k}{2\pi} \int_{\Sigma} d^2z \text{Tr}(A_{\bar{z}} g^{-1} \partial_z g \mp A_z \partial_{\bar{z}} g g^{-1} - A_z A_{\bar{z}} \pm A_{\bar{z}} g^{-1} A_z g), \quad (2.6)$$

where upper (lower) sign refer to vector (axial) gauged model. (The same convention is used in the following.) We call the relation between these two theories as axial-vector duality.

The supersymmetric extension [11] of (2.6) is

$$S_{V/A}(g, \psi_L, \psi_R, A) = S_{V/A}(g, A) + \frac{ik}{2\pi} \int_{\Sigma} d^2z \text{Tr}(\psi_L (\partial_{\bar{z}} \psi_L + [A_{\bar{z}}, \psi_L]) + \psi_R (\partial_z \psi_R \pm [A_z, \psi_R])), \quad (2.7)$$

where  $\psi_L$  and  $\psi_R$  are Weyl fermions with values in  $\text{Lie}(G/H)$ . Here  $\text{Lie}(G/H)$  represents the orthogonal complement of  $\text{Lie}H$  in  $\text{Lie}G$ . The fermions transform under gauge transformation as

$$\begin{aligned} \psi_L &\rightarrow u^{-1} \psi_L u, \\ \psi_R &\rightarrow u^{\mp 1} \psi_R u^{\pm 1}. \end{aligned} \quad (2.8)$$

The conditions for the action (2.7) to have  $N = 2$  supersymmetry is known [12, 13]. First,  $\text{Lie}(G/H)$  must be splitted into two parts,

$$\text{Lie}(G/H) = \tau_+ \oplus \tau_-, \quad (2.9)$$

where  $\tau_+$  and  $\tau_-$  are complex conjugate representations of  $H$ . In addition, the integrability condition,

$$[\tau_+, \tau_+] \subset \tau_+, \quad [\tau_-, \tau_-] \subset \tau_-, \quad (2.10)$$

and the hermiticity condition,

$$\text{Tr}(ab) = 0, \quad \text{for } a, b \in \tau_+ \quad \text{or} \quad a, b \in \tau_-. \quad (2.11)$$

must be satisfied.

We now decompose fermions as  $\psi_{L(R)} = \chi_{L(R)} + \rho_{L(R)}$ , where  $\chi_{L(R)}$  and  $\rho_{L(R)}$  belong to  $\tau_+$  and  $\tau_-$  respectively. By using the equation (2.11), the action (2.7) is written as

$$S_{V/A}(g, \psi_L, \psi_R, A) = S_{V/A}(g, A) + \frac{ik}{2\pi} \int_{\Sigma} d^2z \text{Tr} \left( \chi_L (\partial_{\bar{z}} \rho_L + [A_{\bar{z}}, \rho_L]) + \chi_R (\partial_z \rho_R \pm [A_z, \rho_R]) \right). \quad (2.12)$$

Under the condition (2.10), this action has  $N = (2, 2)$  supersymmetry [13, 14].  $N = 2$  supersymmetry transformation laws for left moving part are given by

$$\begin{aligned} \delta g &= i\epsilon_+(z)g\chi_L + i\epsilon_-(z)g\rho_L, \\ \delta\chi_L &= -\frac{i}{k^2}\epsilon_+(z)\chi_L\chi_L + \epsilon_-(z)P_+ \left[ g^{-1}(\partial_z g \pm A_z g - gA_z) - \frac{i}{k^2}(\chi_L\rho_L + \rho_L\chi_L) \right], \\ \delta\rho_L &= \epsilon_+(z)P_- \left[ g^{-1}(\partial_z g \pm A_z g - gA_z) - \frac{i}{k^2}(\chi_L\rho_L + \rho_L\chi_L) \right] - \frac{i}{k^2}\epsilon_-(z)\rho_L\rho_L, \\ \delta\chi_R &= \delta\rho_R = \delta A_i = 0. \end{aligned} \quad (2.13)$$

For right moving part,

$$\begin{aligned} \delta g &= i\bar{\epsilon}_+(\bar{z})\chi_R g + i\bar{\epsilon}_-(\bar{z})\rho_R g, \\ \delta\chi_R &= \frac{i}{k^2}\bar{\epsilon}_+(\bar{z})\chi_R\chi_R + \bar{\epsilon}_-(\bar{z})P_+ \left[ (\partial_{\bar{z}} g \pm A_{\bar{z}} g - gA_{\bar{z}})g^{-1} + \frac{i}{k^2}(\chi_R\rho_R + \rho_R\chi_R) \right], \\ \delta\rho_R &= \bar{\epsilon}_+(\bar{z})P_- \left[ (\partial_{\bar{z}} g \pm A_{\bar{z}} g - gA_{\bar{z}})g^{-1} + \frac{i}{k^2}(\chi_R\rho_R + \rho_R\chi_R) \right] + \frac{i}{k^2}\bar{\epsilon}_-(\bar{z})\rho_R\rho_R, \\ \delta\chi_L &= \delta\rho_L = \delta A_i = 0. \end{aligned} \quad (2.14)$$

Here  $P_{\pm}$  denote projections to  $\tau_{\pm}$ .

The action (2.12) also has  $U(1)$   $R$  symmetry, whose charge is assigned 1 for  $\chi_{L(R)}$ ,  $-1$  for  $\rho_{L(R)}$ , 0 for  $g$  and  $A_i$ . Note that  $N = 2$  supersymmetry transformations consist of  $\Delta R = 1$  sector and  $\Delta R = -1$  sector.

Let  $G_{\pm}(z)$  be the currents associated with the supersymmetry transformations whose parameters are  $\epsilon_{\pm}(z)$ , and  $J(z)$  be the  $U(1)$  current for left  $R$  symmetry. Then the energy-momentum tensor  $T(z)$ , the supercurrents  $G_+(z)$ ,  $G_-(z)$  and  $U(1)$  current  $J(z)$  generate a  $N = 2$  superconformal algebra. Together with a right moving part, the theory has  $N = (2, 2)$  superconformal symmetry.

The mirror duality is an isomorphism between two  $N = (2, 2)$  superconformal field theories which differ only in the sign of the right moving  $U(1)$  current [15]. This is a trivial relation from a CFT point of view, so it is a symmetry between the two theories. However, target space geometries corresponding to these theories are quite different in general, since the geometrical meaning of the operators changes if the relative sign of right  $U(1)$  charge to left  $U(1)$  charge is flipped. The operator product expansions, especially,

$$J(z)G_{\pm}(w) \sim \pm \frac{G_{\pm}(w)}{z-w} \quad (2.15)$$

show that the change in the sign of the  $U(1)$  current corresponds to the interchange between  $G_+$  and  $G_-$ . It corresponds to a complex conjugation  $\tau_+ \leftrightarrow \tau_-$ . Therefore the mirror transformation is accomplished by taking a complex conjugation only for right moving part.

### 3 The $SU(2)/U(1)$ model

We now show that the vector gauged model is transformed to the axial gauged model by a mirror transformation in the  $SU(2)/U(1)$  model. The generators of the  $SU(2)$  Lie algebra are

$$T = \{T_1, T_2, T_3\}, \quad T_i = \sigma_i/2, \quad (3.1)$$

where  $\sigma_i$  are Pauli matrices. If we gauge  $T_3$  direction,  $\text{Lie}(G/H)$  is spanned by  $\{T_1, T_2\}$ . We can take an almost complex structure  $h$  acting on  $\text{Lie}(G/H)$  as follows,

$$h T_1 = -T_2, \quad h T_2 = T_1. \quad (3.2)$$

With respect to  $h$ ,  $\text{Lie}(G/H)$  is splitted into holomorphic and anti-holomorphic parts whose generators are  $T_+ = T_1 + iT_2$  and  $T_- = T_1 - iT_2$  respectively. Then the integrability condition (2.10) and the hermiticity condition (2.11) are satisfied. In this case, complex conjugation is realized by a following operation,

$$T_{\pm} \rightarrow s T_{\pm} s^{-1} = T_{\mp}, \quad \text{with } s = e^{i\pi T_1} = i\sigma_1. \quad (3.3)$$

The partition function of the  $SU(2)/U(1)$  vector gauged Wess-Zumino-Witten model is

$$Z = \int \mathcal{D}g \mathcal{D}\psi_L \mathcal{D}\psi_R \mathcal{D}A \exp[-S_V(g, \psi_L, \psi_R, A)]. \quad (3.4)$$

As discussed in section 2, the mirror transformation is to take a complex conjugation only for the right moving part, so the mirror transformation of the fermions are given by

$$\begin{aligned} \psi_L &\rightarrow \tilde{\psi}_L = \psi_L, \\ \psi_R &\rightarrow \tilde{\psi}_R = s\psi_R s^{-1}. \end{aligned} \quad (3.5)$$

From the requirement of the supersymmetry, the mirror transformation of the field  $g$  is determined as

$$g \rightarrow \tilde{g} = sg. \quad (3.6)$$

The gauge fields  $A_i$  are invariant under the mirror transformation.

Thus the vector gauged model is transformed to

$$Z_m = \int \mathcal{D}\tilde{g} \mathcal{D}\tilde{\psi}_L \mathcal{D}\tilde{\psi}_R \mathcal{D}A \exp[-S_V(\tilde{g}, \tilde{\psi}_L, \tilde{\psi}_R, A)], \quad (3.7)$$

by the mirror transformation. The classical action is invariant under the vector gauge transformation,

$$\begin{aligned} \tilde{g} &\rightarrow u^{-1}\tilde{g}u, \\ \tilde{\psi}_L &\rightarrow u^{-1}\tilde{\psi}_L u, \\ \tilde{\psi}_R &\rightarrow u^{-1}\tilde{\psi}_R u, \\ A_i &\rightarrow A_i + u^{-1}\partial_i u, \end{aligned} \quad (3.8)$$

with  $u = \exp(i\epsilon T_3)$ .

We now change the variables from  $\tilde{g}, \tilde{\psi}_L, \tilde{\psi}_R$  to  $g, \psi_L, \psi_R$ . By using the relation  $sA_i s^{-1} = -A_i$ , which follows from the fact that the gauge fields are proportional to  $T_3$ , we can show that

$$S_V(\tilde{g}, \tilde{\psi}_L, \tilde{\psi}_R, A) = S_A(g, \psi_L, \psi_R, A) - \frac{k}{2\pi} \int d^2z \text{Tr}(i\pi T_1 F_{z\bar{z}}). \quad (3.9)$$

Here  $F_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z$  is also proportional to  $T_3$ . The last term, which comes from the “classical” anomaly of the bosonic part of the action, vanishes since  $\text{Tr}(T_1 T_3) = 0$ . Therefore, after change of variables, the classical action of the mirror transformed model (3.7) becomes that of the axial gauged model.

Corresponding to this, the gauge transformation laws for the fields  $g, \psi_L, \psi_R$  can be read off from (3.8). The result is the axial gauge transformation,

$$\begin{aligned} g &\rightarrow ugu, \\ \psi_L &\rightarrow u^{-1} \psi_L u, \\ \psi_R &\rightarrow u \psi_R u^{-1}. \end{aligned} \quad (3.10)$$

Here we have used the relation  $sus^{-1} = u^{-1}$ .

The Jacobian associated with the change of variables from  $\tilde{\psi}_L, \tilde{\psi}_R$  to  $\psi_L, \psi_R$  can be calculated by the chiral anomaly,

$$\mathcal{D}\tilde{\psi}_L \mathcal{D}\tilde{\psi}_R = \mathcal{D}\psi_L \mathcal{D}\psi_R \exp \left[ -\frac{1}{4\pi} \int d^2z \text{Tr}(i\pi T_1 F_{z\bar{z}}) \right]. \quad (3.11)$$

Here “Tr” represents the trace in the adjoint representation of  $G$ . Again, the integrand vanishes since  $F_{z\bar{z}}$  is proportional to  $T_3$ . As for  $\mathcal{D}g$ , we can take a Haar measure, so the integration measures are invariant under change of variables.

Thus we have shown that the vector gauged model (3.4) is transformed to the axial gauged model,

$$Z_m = \int \mathcal{D}g \mathcal{D}\psi_L \mathcal{D}\psi_R \mathcal{D}A \exp[-S_A(g, \psi_L, \psi_R, A)], \quad (3.12)$$

by the mirror transformation. Therefore the  $SU(2)/U(1)$  vector gauged model is equivalent to the  $SU(2)/U(1)$  axial gauged model as  $N = (2, 2)$  superconformal field theories.

## 4 The $SL(2, R)/U(1)$ model

Next, we apply the method explained in the last section to the  $SL(2, R)/U(1)$  case, which is more interesting physically since it corresponds to a black hole [3]. The generators of the  $SL(2, R)$  Lie algebra are

$$T = \{T_1, T_2, T_3\}, \quad T_1 = \frac{\sigma_1}{2}, \quad T_2 = \frac{i\sigma_2}{2}, \quad T_3 = \frac{\sigma_3}{2}. \quad (4.1)$$

If we gauge the  $T_3$  direction, it is known that the vector gauged model and the axial gauged model are self-dual, and correspond to a Lorentzian black hole [16]. In this case, we cannot

take a complex structure which satisfies the hermiticity condition (2.11). Instead, we take an almost product structure. The possibility of a theory with an almost product structure to have  $N = 2$  supersymmetry is discussed in ref. [17]. An almost product structure  $\Pi$  is a tensor that maps  $\text{Lie}(G/H)$  to  $\text{Lie}(G/H)$  and satisfies  $\Pi^2 = 1$ . Since the eigenvalue of  $\Pi$  is  $\pm 1$ ,  $\text{Lie}(G/H)$  is decomposed into two parts  $\tau_{\pm}$  according to their eigenvalues. In the present case, we take an almost product structure as follows,

$$\Pi T_1 = T_2, \quad \Pi T_2 = T_1. \quad (4.2)$$

The generators of  $\tau_{\pm}$  are  $T_{\pm} = T_2 \pm T_1$ . It satisfies the integrability condition and the hermiticity condition which are defined in the same way as (2.10) and (2.11). This makes the theory to have  $N = 2$  supersymmetry. The discussion about the mirror transformation is the same as in the  $SU(2)/U(1)$  case with a matrix  $s$  be replaced by  $s = \exp(\pi T_2) = i\sigma_2$ . In this case, the mirror partners are self-dual to each other, so the equivalence of the two models under mirror duality is manifest.

Next we consider the most interesting case, that is, the case of gauging the  $T_2$  direction. The corresponding background is “trumpet” for the vector gauged model, and “cigar” for the axial gauged model [16]. These backgrounds are called Euclidean black holes. As we will see later, the “trumpet” background has a singularity, while the “cigar” background has no singularity.

In this case, we take a complex structure acting on  $\text{Lie}(G/H)$  as

$$h T_1 = -T_3, \quad h T_3 = T_1, \quad (4.3)$$

then  $\text{Lie}(G/H)$  is splitted into  $\tau_{\pm}$  whose generators are  $T_{\pm} = T_1 \pm iT_3$ . Complex conjugation is realized as follows,

$$T_{\pm} \rightarrow s T_{\pm} s^{-1} = T_{\mp}, \quad \text{with} \quad s = ie^{-i\pi T_1} = \sigma_1. \quad (4.4)$$

The mirror transformations of  $SL(2, R)$  valued field  $g$  and  $\text{Lie}(SL(2, R)/U(1))$  valued fields  $\psi_L$  and  $\psi_R$  are given by

$$\begin{aligned} g &\rightarrow \tilde{g} = sg = \sigma_1 g, \\ \psi_L &\rightarrow \tilde{\psi}_L = \psi_L, \\ \psi_R &\rightarrow \tilde{\psi}_R = s\psi_R s^{-1} = e^{-i\pi T_1} \psi_R e^{i\pi T_1}. \end{aligned} \quad (4.5)$$

There is a subtlety in this case. That is, the determinant of  $\tilde{g}$  is  $-1$ , so  $\tilde{g}$  no longer belongs to  $SL(2, R)^*$ . So we must carefully analyze the mirror transformation.

We first consider the classical action. We start from the action of the vector gauged Wess-Zumino-Witten model,

$$S_V(g, \psi_L, \psi_R, A). \quad (4.6)$$

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\*This is not because the choice of  $s$  is not appropriate. Although other choices of  $s$  are possible to realize the mirror transformation, the field  $\tilde{g} = sg$  does not belong to  $SL(2, R)$  anyway.

After the mirror transformation, we obtain the action

$$S_V(\tilde{g}, \tilde{\psi}_L, \tilde{\psi}_R, A). \quad (4.7)$$

Here  $\tilde{g} = sg = \sigma_1 g$  is not  $SL(2, R)$  valued, so the action (4.7) is not well-defined as a  $SL(2, R)/U(1)$  vector gauged Wess-Zumino-Witten model. However, this is not an assertion that the action (4.7) is not well-defined in any sense. In fact, after change of variables from  $\tilde{g}, \tilde{\psi}_L, \tilde{\psi}_R$  to  $g, \psi_L, \psi_R$ , the action is written as

$$S_A(g, \psi_L, \psi_R, A), \quad (4.8)$$

as in the  $SU(2)/U(1)$  case. Here  $g$  is  $SL(2, R)$  valued, so this action is well-defined as a  $SL(2, R)/U(1)$  axial gauged Wess-Zumino-Witten model. Hence, the action (4.7), which is related to (4.8) by change of variables, is also well-defined in this sense.

We next consider the integration measures. The Jacobian associated with the change of variables from  $\tilde{\psi}_L, \tilde{\psi}_R$  to  $\psi_L, \psi_R$  can be calculated as in the  $SU(2)/U(1)$  case, and we can see that  $\mathcal{D}\tilde{\psi}_L \mathcal{D}\tilde{\psi}_R = \mathcal{D}\psi_L \mathcal{D}\psi_R$ . As for  $\mathcal{D}\tilde{g}$ , we can not take a Haar measure for  $SL(2, R)$  since  $\det \tilde{g} = -1$ . Instead, we take a Haar measure for  $GL(2, R)$ . In the parameterization  $\tilde{g} = (\tilde{g}_{ij}) \in GL(2, R)$ , the Haar measure for  $GL(2, R)$  is given by

$$d\tilde{g} = |\det \tilde{g}|^{-2} d\tilde{g}_{11} d\tilde{g}_{12} d\tilde{g}_{21} d\tilde{g}_{22}. \quad (4.9)$$

We are now considering the case of  $\det \tilde{g} = -1$ , so  $d\tilde{g} = d\tilde{g}_{11} d\tilde{g}_{12} d\tilde{g}_{21} d\tilde{g}_{22}$ . After change of variables from  $\tilde{g}$  to  $g = s^{-1}\tilde{g} = \sigma_1 \tilde{g}$ , it is written as  $d\tilde{g} = dg_{11} dg_{12} dg_{21} dg_{22}$ , which is nothing but the Haar measure for  $SL(2, R)$ .

Thus, by the change of variables, the mirror transformed model

$$Z_m = \int \mathcal{D}\tilde{g} \mathcal{D}\tilde{\psi}_L \mathcal{D}\tilde{\psi}_R \mathcal{D}A \exp[-S_V(\tilde{g}, \tilde{\psi}_L, \tilde{\psi}_R, A)], \quad (4.10)$$

is rewritten as the  $SL(2, R)/U(1)$  axial gauged Wess-Zumino-Witten model,

$$Z_m = \int \mathcal{D}g \mathcal{D}\psi_L \mathcal{D}\psi_R \mathcal{D}A \exp[-S_A(g, \psi_L, \psi_R, A)]. \quad (4.11)$$

Therefore the vector gauged model which corresponds to singular “trumpet” background and the axial gauged model which corresponds to non-singular “cigar” background are mirror partner, and hence they are equivalent as  $N = (2, 2)$  superconformal field theories.

We now see the reason why the mirror duality relates the background with a singularity to that without a singularity. Space-time singularities in gauged Wess-Zumino-Witten model occur at fixed points of the gauge transformation. We parameterize a  $SL(2, R)$  valued field  $g$  as

$$g = e^{\frac{i}{2}\theta_L \sigma_2} e^{\frac{1}{2}r \sigma_1} e^{\frac{i}{2}\theta_R \sigma_2}, \quad (4.12)$$

with  $0 \leq r < \infty$ ,  $0 \leq \theta_L < 2\pi$ ,  $-2\pi \leq \theta_R < 2\pi$ . Then the vector gauge transformation  $g \rightarrow u^{-1}gu$  with  $u = \exp(i\epsilon\sigma_2/2)$  of the original model has a fixed point at  $r = 0$ . On the other hand, the vector gauge transformation  $\tilde{g} \rightarrow u^{-1}\tilde{g}u$  of the mirror transformed model has no fixed



point. This originates from the peculiar property of the mirror transformation, that is,  $\det \tilde{g} = -1$ .

Note that a similar situation takes place in bosonic three-dimensional black holes [18]. In that case, a black string [19] based on  $SL(2, R) \times U(1)/U(1)$  gauged Wess-Zumino-Witten model, which has a singularity, is dual to a black hole without a curvature singularity [20]. However, the dual model corresponds to an orbifold of a  $SL(2, R)$  Wess-Zumino-Witten model with the  $SL(2, R)$  valued field replaced by a field with negative determinant. It is interesting to investigate the relation of this duality to mirror symmetry.

## 5 Discussion

We have shown that the vector gauged model corresponding to a singular background and the axial gauged model corresponding to a non-singular background are mirror partner. We now discuss the structure of the singularity which is expected from the equivalence of the two models under mirror duality.

In ref. [7], it is shown that the singularity in  $SL(2, R)/U(1)$  Lorentzian black hole is described by a topological field theory and has a structure of chiral ring. This is based on a twisted version of the  $N = 2$  gauged Wess-Zumino-Witten model [13]. By the twisting procedure [21], one of the supercharges  $G_{\pm}$  becomes a BRS operator which generates a topological symmetry. The chiral ring of the model is calculated by a three point function of BRS invariant operators. The crucial observation is that the only contribution to the path integral comes from a fixed point of the BRS transformation, and furthermore it coincides to the space-time singularity. It is also speculated that BRS fixed points may generally occur at space-time singularities.

We now investigate whether this is indeed the case for the  $SL(2, R)/U(1)$  Euclidean black holes. We twist the vector gauged model by  $T(z) \rightarrow T(z) - \partial_z J(z)/2$ ,  $\bar{T}(\bar{z}) \rightarrow \bar{T}(\bar{z}) + \partial_{\bar{z}} \bar{J}(\bar{z})/2$ . By this procedure, BRS operator is defined as  $Q_B = G_+ + \bar{G}_-$ . The BRS transformations of the fields can be read off from (2.13) and (2.14). By using the fact that  $\chi_{L(R)}$  is proportional to  $\sigma_1 + i\sigma_3$ , and  $\rho_{L(R)}$  is proportional to  $\sigma_1 - i\sigma_3$ , the transformation laws are written as <sup>†</sup>

$$\begin{aligned}\delta_B g &= i\epsilon(g\chi_L + \rho_R g), \\ \delta_B \rho_L &= \epsilon P_- [g^{-1}(\partial_z g + A_z g - g A_z)], \\ \delta_B \chi_R &= \epsilon P_+ [(\partial_{\bar{z}} g + A_{\bar{z}} g - g A_{\bar{z}})g^{-1}], \\ \delta_B \chi_L &= \delta_B \rho_R = \delta A_i = 0.\end{aligned}\tag{5.1}$$

In the parameterization

$$g = \begin{pmatrix} a & u \\ -v & b \end{pmatrix}, \quad \text{with } ab + uv = 1,\tag{5.2}$$

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<sup>†</sup>Here we use the same notations for the fields as in the untwisted model although their conformal weights are shifted by twisting.

the BRS fixed points for bosonic variables are given by the following conditions

$$P\partial_z R - R(\partial_z P + ia_z P) = 0, \quad \text{and} \quad Q\partial_{\bar{z}} R - R(\partial_{\bar{z}} Q - ia_{\bar{z}} Q) = 0, \quad (5.3)$$

with  $P = a - b - i(u - v)$ ,  $Q = a - b + i(u - v)$ ,  $R = a + b - i(u + v)$ . Here we put  $A_i = a_i T_2$ . There are several branches of possible solutions. The first branch is  $P = Q = 0$ . In the parameterization (4.12),  $P$  and  $Q$  are written as

$$P = -2i \sinh \frac{r}{2} e^{\frac{i}{2}(\theta_L - \theta_R)}, \quad Q = 2i \sinh \frac{r}{2} e^{-\frac{i}{2}(\theta_L - \theta_R)}. \quad (5.4)$$

Therefore the solution of this branch is  $r = 0$ . The second branch of possible solutions of (5.3) is  $R = 0$ . However, there is no solution in this branch since

$$R = 2 \cosh \frac{r}{2} e^{-\frac{i}{2}(\theta_L + \theta_R)} \quad (5.5)$$

never vanishes. Thus the BRS fixed point occurs at  $r = 0$ , just at which a singularity lies.

Next we study BRS fixed points of the axial gauged model. Since it is a mirror partner of the vector gauged model, we must twist as  $T(z) \rightarrow T(z) - \partial_z J(z)/2$ ,  $\bar{T}(\bar{z}) \rightarrow \bar{T}(\bar{z}) - \partial_{\bar{z}} \bar{J}(\bar{z})/2$  in order to compare the results with that of the vector gauged model twisted as in the last paragraph. In this case, the BRS operator is defined as  $Q_B = G_+ + \bar{G}_+$ . Then the BRS transformation laws are

$$\begin{aligned} \delta_B g &= i\epsilon(g\chi_L + \chi_R g), \\ \delta_B \rho_L &= \epsilon P_- [g^{-1}(\partial_z g - A_z g - g A_z)], \\ \delta_B \rho_R &= \epsilon P_- [(\partial_{\bar{z}} g - A_{\bar{z}} g - g A_{\bar{z}})g^{-1}], \\ \delta_B \chi_L &= \delta_B \chi_R = \delta A_i = 0. \end{aligned} \quad (5.6)$$

The BRS fixed points are given by

$$R\partial_z P - P(\partial_z R + ia_z R) = 0, \quad \text{and} \quad S\partial_{\bar{z}} P - P(\partial_{\bar{z}} S - ia_{\bar{z}} S) = 0, \quad (5.7)$$

with  $S = a + b + i(u + v)$ . The first branch of possible solutions is  $R = S = 0$ . However, there is no solution in this branch since  $R$  and

$$S = 2 \cosh \frac{r}{2} e^{\frac{i}{2}(\theta_L + \theta_R)} \quad (5.8)$$

never vanish. The second branch is  $P = 0$ , and it corresponds to  $r = 0$ . However, this branch does not contribute to the calculation of chiral ring by the following reason. We can see that  $P$  is a BRS invariant operator by analyzing  $\delta_B g$ . Since the chiral ring is calculated by a three point function of BRS invariant operators, the integrand vanishes just at which the contribution to the path integral comes.

These results are consistent with the speculation that BRS fixed points may occur at space-time singularities. However, it seems that this leads to an asymmetry in the chiral ring between mirror partners. This is inconsistent with the result that the singular ‘‘trumpet’’

background and the non-singular “cigar” background are mirror partners, and hence equivalent as  $N = (2, 2)$  superconformal field theories. This is a puzzle I mentioned in introduction.

Exactly speaking, however, there is another possibility to solve the equations (5.3) or (5.7). For the vector gauged model, it is given by the following equations,

$$P = fR \quad \text{with} \quad \partial_z f + ia_z f = 0, \quad \text{and} \quad Q = hR \quad \text{with} \quad \partial_{\bar{z}} h - ia_{\bar{z}} h = 0. \quad (5.9)$$

We note that this branch does not consist of classical solutions of the gauged Wess-Zumino-Witten model, so the contributions would vanish exponentially for  $k \rightarrow \infty$  [13]. A similar possibility is also allowed for the axial gauged model. The contribution from this branch, which can be interpreted as a quantum correction to the structure of the singularity, may compensate for the apparent asymmetry in the chiral ring between mirror partners.

Note that a similar situation occurs in a discussion of topology change of Calabi-Yau manifolds [22]. In the course of topology change, target manifolds necessarily go through a singular configuration, while the mirror partner does not. In that case, non-perturbative world-sheet instanton corrections [23] smooth a discreteness associated with the singularity, and mirror symmetry is preserved.

To make a definite statement, we must further investigate a three point function of BRS invariant operators, especially the contribution from the third branch.

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